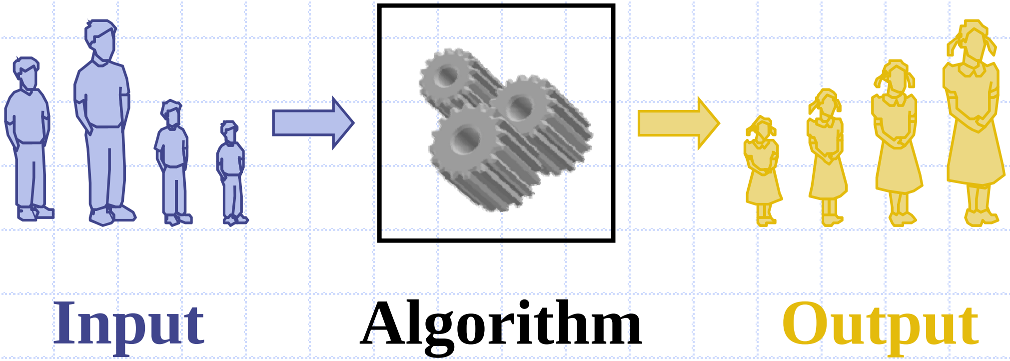
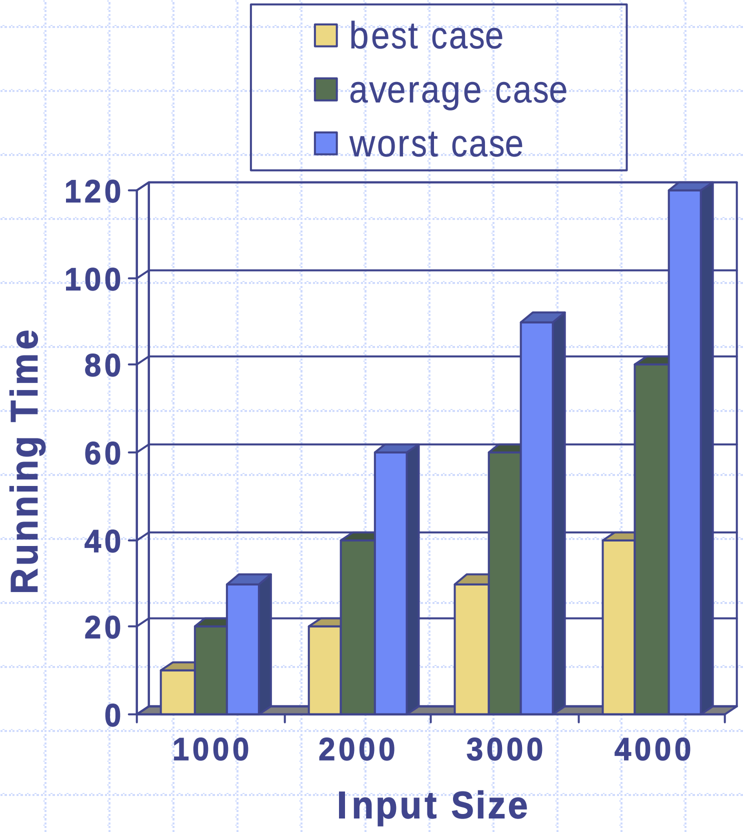
Analysis of Algorithms



An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

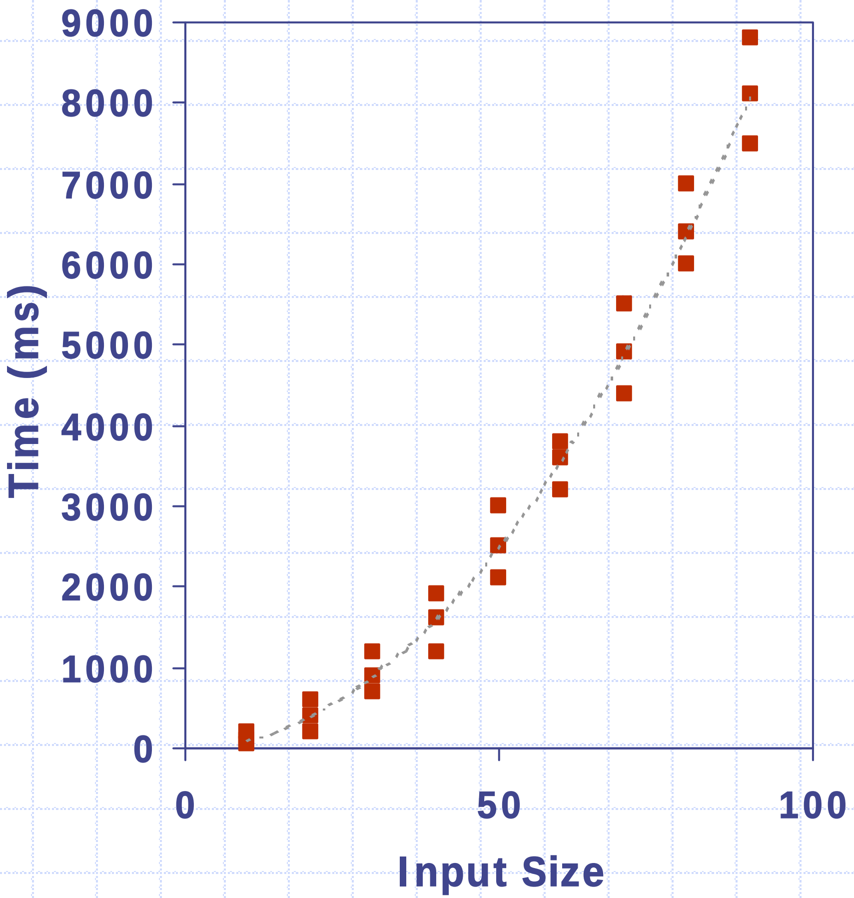
Running Time (§3.1)

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Experimental Studies (§ 3.1.1)

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a function, like the built-in clock() function, to get an accurate measure of the actual running time
- Plot the results



Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode (§3.1.2)

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

Algorithm *arrayMax*(A, n)
  Input array A of n integers
  Output maximum element of A

  $currentMax \leftarrow A[0]$
  for $i \leftarrow 1$ **to** $n - 1$ **do**
    if $A[i] > currentMax$ **then**
      $currentMax \leftarrow A[i]$
  return $currentMax$

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …

- Method/Function call

  var.method (*arg* [, *arg*…])

- Return value

  return *expression*

- Expressions

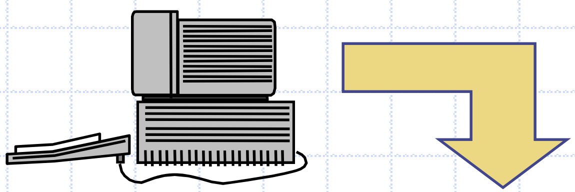
  $\leftarrow$ Assignment (like $=$ in C++)

  $=$ Equality testing (like $==$ in C++)

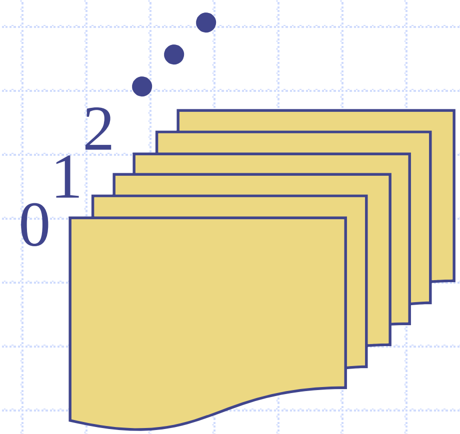
  n^2 Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

- A **CPU**



- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character



- Memory cells are numbered and accessing any cell in memory takes unit time.

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method
  - Comparing two numbers

Counting Primitive Operations (§3.4.1)

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm *arrayMax*(A, n)	# operations
$currentMax \leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	$1 + n$
if $A[i] > currentMax$ **then**	$2(n - 1)$
$currentMax \leftarrow A[i]$	$2(n - 1)$
{ increment counter i }	$2(n - 1)$
return $currentMax$	1
Total	$7n - 2$

Estimating Running Time

- Algorithm ***arrayMax*** executes $7\boldsymbol{n} - 2$ primitive operations in the worst case.
- What about in the best case ?
- Define:

  $\boldsymbol{a}$ = Time taken by the fastest primitive operation

  $\boldsymbol{b}$ = Time taken by the slowest primitive operation
- Let $\boldsymbol{T}(\boldsymbol{n})$ be worst-case time of ***arrayMax.*** Then

$$\boldsymbol{a}\,(7\boldsymbol{n} - 2) \leq \boldsymbol{T}(\boldsymbol{n}) \leq \boldsymbol{b}(7\boldsymbol{n} - 2)$$

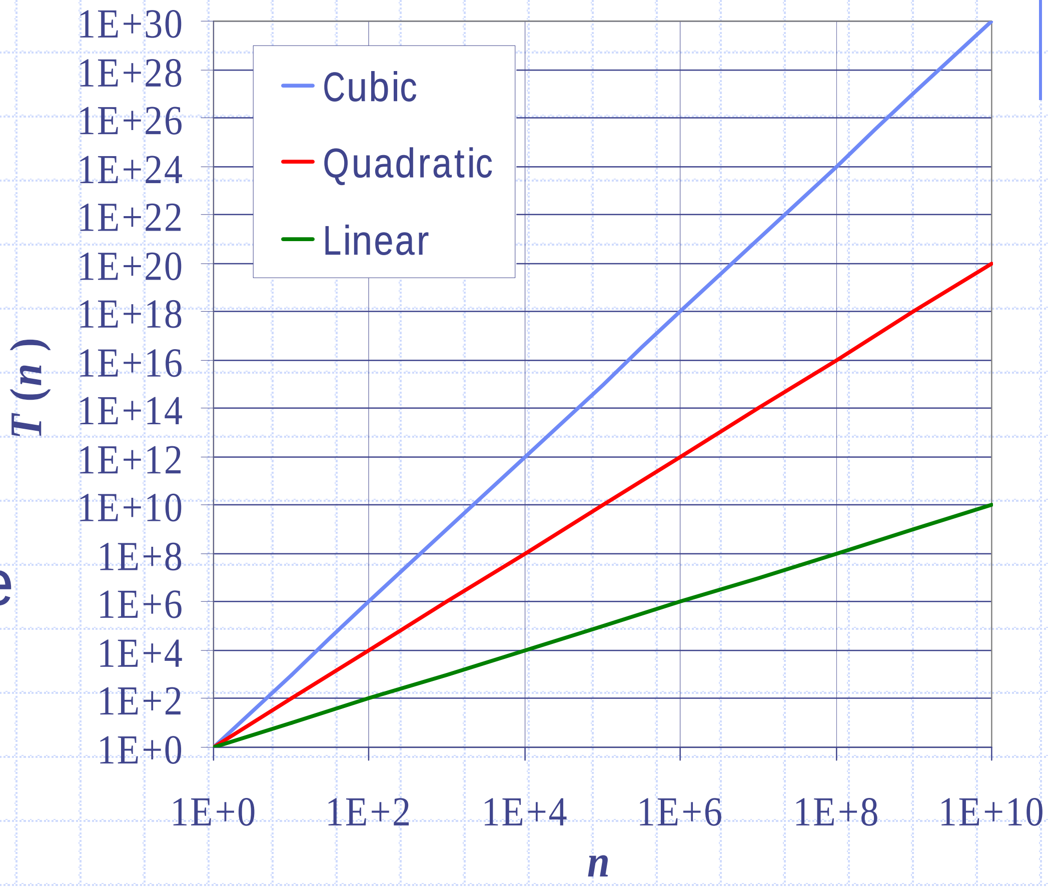
- Hence, the running time $\boldsymbol{T}(\boldsymbol{n})$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

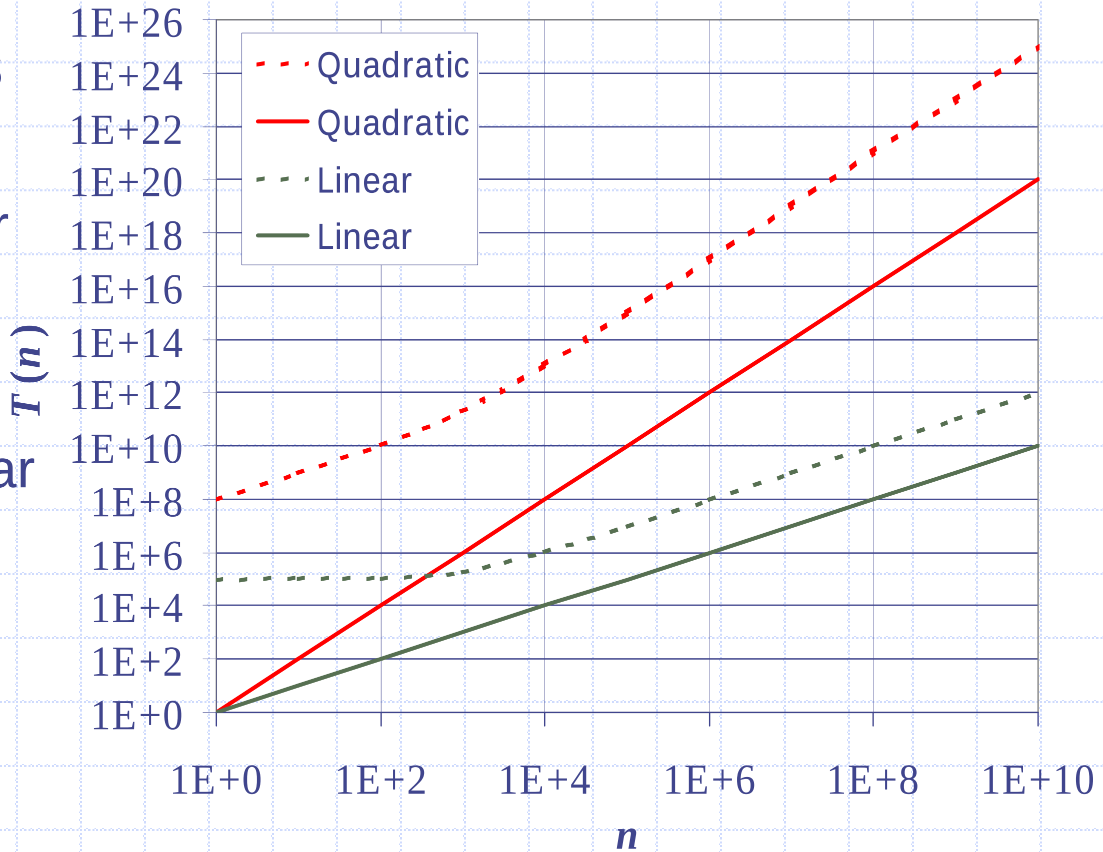
Growth Rates

- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function



Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Big-Oh Notation (§3.5)

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

$$f(n) \leq cg(n) \text{ for } n \geq n_0$$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - Pick $c = 12$ and $n_0 = 1$

Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant

More Big-Oh Examples

- $7n - 2$

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 28$ and $n_0 = 1$

- $3 \log n + \log \log n$

  $3 \log n + \log \log n$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + \log \log n \leq c \bullet \log n$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that
    $$f(n) \geq c \cdot g(n) \text{ for } n \geq n_0$$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that
    $$c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n) \text{ for } n \geq n_0$$

Intuition for Asymptotic Notation

Big-Oh

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$